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QUASI-ASSOCIATIVITY AND FLATNESS CRITERIA FOR QUADRATIC ALGEBRA DEFORMATIONS

BY

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ABSTRACT

Let R_h be the quantum R-matrix corresponding to a Drinfeld–Jimbo quantum group $U_h(\mathcal{G})$. Suppose a finite dimensional representation M_h of $U_h(\mathcal{G})$ is given. Then R_h induces an operator on $M_h^{\otimes 2}$ and S_h, its composition with the standard transposition, is the Yang–Baxter operator. It turns out that the space $M_h^{\otimes 2}$ admits the decomposition $M_h = \bigoplus_i^n J_{ih}$ where J_{ih} are the eigensubspaces of S_h. Consider the quadratic algebras (M_h, E_h^k) where $E_h^k = \bigoplus_{i \neq k} J_{ih}$. We prove that all (M_h, E_h^k) are flat deformations of the quadratic algebras (V_0, E_0^k). Let $\mathrm{End}(M_h; J_{1h}, ..., J_{nh})$ be the quantum semigroup corresponding to this decomposition. Our second result is that this gives a flat deformation of the quantum semigroup $\mathrm{End}(M_0; J_{1,0}, \dots, J_{n,0})$.

Introduction

One of the important operators appearing in the theory of quantum groups is the Yang–Baxter operator, S_h, also called the quantum symmetry constraint. It is given by composition of the standard transposition with the quantum R-matrix, R_h. If M_h, N_h are representations of the quantized universal enveloping (QUE) algebra, $U_h(\mathcal{G})$, which are, respectively, deformations of the representations M and N of $U(\mathcal{G})$, then S_h defines an intertwining operator between the representation $M_h \otimes N_h$ and $N_h \otimes M_h$. We are interested in the

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case when $M = N$ with M an arbitrary, not necessarily irreducible, representation. As we explain in §1, S_h is a diagonalizable element of $\mathrm{End}(M_h \otimes M_h)$. For the study of quantum spaces we consider λ_h, an eigenvalue of S_h acting on $M_h \otimes M_h$, and define the subspace $E_{\lambda_h} = \mathrm{Im}(S_h - \lambda_h) \subset M_h \otimes M_h$ which is complementary to the λ_h eigenspace of S_h. For the case of quantum semigroups we consider the representation of $U_h(\mathcal{G})$ on $\mathrm{End}(M_h)$ and the subspace $E_{h,\mathrm{ad}} = \mathrm{Im}([S_h, \cdot]) \subset \mathrm{End}(M_h \otimes M_h) \cong \mathrm{End}(M_h) \otimes \mathrm{End}(M_h)$.

The quadratic algebra $\{M, E\}$ is, by definition, the quotient of the tensor algebra by the ideal generated by the "quadratic" subspace E of $M \otimes M$. In this paper we study the h dependent families of quadratic algebras $\{M_h, E_{\lambda_h}\}$ and $\{\mathrm{End}(M_h), E_{h,\mathrm{ad}}\}$. The first family is generated by a quadratic subspace complementary to the λ_h eigenspace of S_h and so depends on the choice of the eigenvalue. The first example gives us the function algebra of a quantum space. The second example is more canonical, being the quotient of the tensor algebra of $\mathrm{End}(M_h)$ by the ideal generated by the subspace

$$\mathrm{Im}[S_h, \cdot] = \mathrm{span}\{S_h \circ X \otimes Y - X \otimes Y \circ S_h\}$$

in $\mathrm{End}(M_h) \otimes \mathrm{End}(M_h)$. This is the standard Fade'ev–Reshetikhin–Takhtajan construction of the quantum semigroup corresponding to the Yang–Baxter operator S_h.

The basic question which concerns us is whether the dimension of the homogeneous components is constant in h near 0. When this occurs the family is said to be flat and the parametrized family is called a deformation of the quadratic algebra at $h = 0$.

A necessary condition for flatness is clearly that the space E_{λ_h} have constant dimension. The limit of S_h at $h = 0$ is the classical transposition, $\sigma(u \otimes v) = v \otimes u$ with eigenvalues ± 1 and eigenspaces, the symmetric or skew symmetric two tensors, $E_\pm$. When there are more than two eigenvalues of S_h for $h \neq 0$ the dimension of the eigenspace for some λ_h goes up at $h = 0$ and, as a consequence, the dimension of E_{λ_h} will drop. In this case we replace $E_\pm$ with the subspace $\lim_{h\to 0} E_{\lambda_h}$.

Our first result, Theorem 1.1, states that for any eigenvalue the resulting family of quadratic algebras is flat. Moreover one can identify the coefficient, λ, of the linear term in the power series expansion in h of $\lambda_h = \pm 1 + h\lambda + O(h^2)$, as an eigenvalue of the polarized Casimir operator, $\mathbf{t}$, acting on $M \otimes M$. The limiting subspace defined above, denoted $E_{\lambda,\pm}$, is a complement to the intersection of the ± 1 eigenspace of σ and the λ eigenspace of t. Stated differently: For an arbitrary representation M, a choice of the symmetric or the skew symmetric subspace of $M \otimes M$ and an eigenvalue λ of t on that subspace, the quadratic algebra $\{M_h, E_{\lambda_h}\}$ determines a (flat) deformation of the quadratic algebra $\{M, E_{\lambda,\pm}\}$.

Our second result, Theorem 4.1, states that the quantum semigroup corresponding to the Yang–Baxter operator, S_h, determines a (flat) deformation of the semigroup $\{A \in \operatorname{End}(M) | [A \otimes A, t] = 0\}$. This leads us to some interesting new examples which deserve further study.

In the remarks following the statement of Theorem 1.1 and a further remark following its proof in §3 we explain why the analogous result to Theorem 1.1 is not true when E_{λ_h} is replaced by $\operatorname{Im}(S_h - \lambda_h) \cap \cdots \cap \operatorname{Im}(S_h - \lambda'_h)$, the complement to the sum of two or more eigenspaces of S_h.

The proof of Theorem 1.1 is based on the existence of an equivalence of the braided monoidal categories between the category of representations of the Drinfeld–Jimbo QUE algebra, $U_h(\mathcal{G})$, and the category of representations of Drinfeld's quasi-Hopf deformation. The latter will be denoted $U(\mathcal{G})_\Phi$, where Φ stands for the (non-trivial) associativity constraint. The quantum symmetry in the category of representations of $U(\mathcal{G})_\Phi$ is transposition composed with e^{ht}. Using this instead of S_h gives rise to a family of quadratic algebras, which are not associative, but the nonassociativity is controlled by the associativity constraint given by the action of Φ; see Proposition 2.1. The equivalence of categories determines a linear isomorphism between this nonassociative quadratic algebra and the corresponding associative quadratic algebras defined using S_h. Flatness for the family of nonassociative quadratic algebras can be proved by elementary arguments.

The paper is organized as follows: Section 1 contains the basic definitions and the statement of the main theorem. Section 2 introduces the notion of monoidal categories "fibered over a category of A-modules" and defines the "quasi-associative tensor algebra" generated by an object in A. Proposition 2.2 describes a natural A module isomorphism between quasi-associative tensor algebras which arises from an equivalence of fibered monoidal categories. Section 3 specializes the discussion to the case of the Drinfeld–Jimbo QUE algebra and Drinfeld's quasi-Hopf QUE algebra. Proposition 3.1 establishes the flatness of the quasi-associative quadratic algebra and, by the isomorphism theorem of section 2, this proves the main theorem. In section 4 we prove the main theorem for quantum semigroups. In section 6 we consider a number of examples and, in particular, apply these methods to study the flatness of what Manin [9] has called quantum semigroups, that is, the Fade'ev–Reshetikhin–Takhtajan type quantization of semigroups of endomorphisms, conformal orthogonal endomorphisms, and the conformal symplectic endomorphisms, [6].

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1. Some background and statement of the main theorem

Our arguments involve formal algebraic calculations and therefore it will be convenient to replace the family of algebras depending on a numerical parameter h with algebras defined over the ring of formal power series, $\mathbf{C}[[h]]$. We return to the case of dependence on a numerical parameter at the end of this section.

Let M be a module over a commutative ring A. For our applications A will be either the complex field, $\mathbf{C}$, or the formal power series ring, $\mathbf{C}[[h]]$. We use $M \otimes M$ or $M^{\otimes 2}$ to denote the tensor product over A, $M \otimes_A M$, and similarly for higher tensor powers. When A is the formal power series ring, $M \otimes M$ will denote the completion of the tensor product in the Krull topology. For any vector space M, $M[[h]]$ will denote the formal power series with coefficients in M. With this convention we can identity $M^{\otimes n}[[h]]$ and $(M[[h]])^{\otimes n}$. The full tensor algebra will be denoted

$$\bigotimes M = \bigoplus_{n=1}^{\infty} M^{\otimes n}.$$

Following Manin [9], for any $E \subset M \otimes M$, we define the quadratic algebra $\{M, E\}$ to be the quotient of the tensor algebra by the ideal I_E generated by the quadratic component, E, that is, the component I_E in degree n is

$$I_E^{(n)} = E \otimes M^{\otimes n-2} + M \otimes E \otimes M^{\otimes n-3} + \cdots + M^{\otimes n-2} \otimes E,$$

and the quadratic algebra is

$$\{M, E\} = (\bigotimes M)/I_E.$$

A **formal deformation** of an algebra B over $\mathbf{C}$ is an algebra B_h over the formal power series ring $\mathbf{C}[[h]]$ satisfying the conditions:

(1) As an $\mathbf{C}[[h]]$ module, $B_h \cong B[[h]]$,

(2) $B_h/hB_h \cong B$ as **C** algebras.

We refer to the first condition as **formal flatness.**

A **quadratic deformation** of a quadratic algebra $\{M, E\}$ is a deformation that is itself a quadratic algebra over $\mathbf{C}[[h]]$ of the type $\{M_h, E_h\}$ where $M_h \cong M[[h]]$ and $E_h \cong E[[h]]$ as $\mathbf{C}[[h]]$ modules.

Let $U_h(\mathcal{G})$ be the well-known Drinfeld–Jimbo QUE algebra, for $\mathcal{G}$ a simple Lie algebra over **C**, [4][6], with multiplication μ_h, comultiplication Δ_h and R-matrix $\mathcal{R}_h$. Since $U(\mathcal{G})$ is rigid as an algebra, there is a $\mathbf{C}$[[h]] algebra automorphism congruent to the identity mod h, ξ_h, such that

$$\mu_h \circ (\xi_h \otimes \xi_h) = \xi_h \circ \mu,$$

where μ is the standard (undeformed) multiplication on $U(\mathcal{G})$ extended $\mathbf{C}[[h]]$ linearly. We shall modify the presentation of $U_h(\mathcal{G})$ by conjugating the operations by ξ_h and denote the deformation in this new presentation by $U'_h(\mathcal{G})$:

(1.1)

$$(U(\mathcal{G})[[h]], \mu = \xi_h^{-1} \circ \mu_h(\xi_h \circ \xi_h), \Delta'_h = (\xi_h^{-1} \otimes \xi_h^{-1}) \circ \Delta_h \circ \xi_h, \mathcal{R}'_h = (\xi_h^{-1} \otimes \xi_h^{-1})(\mathcal{R}_h)).$$

The correspondence between the representations of $U_h(\mathcal{G})$ and $U'_h(\mathcal{G})$ is given by composition with ξ_h; if ρ_h is a representation of $U_h(\mathcal{G})$ then $\rho'_h = \rho_h \circ \xi_h$ is a representation of $U'_h(\mathcal{G})$, and conversely. Representing $\mathcal{R}'_h$ by $\rho'_h \otimes \rho'_h$ gives the same result as representing $\mathcal{R}_h$ by $\rho_h \otimes \rho_h$ and thus the Yang–Baxter operators related to $U_h(\mathcal{G})$ and $U'_h(\mathcal{G})$ are the same.

Henceforth, when referring to the Drinfeld–Jimbo QUE algbra we mean $U'_h(\mathcal{G})$.

Let (M, ρ) be a representation of $U(\mathcal{G})$. Let ρ_h be the $\mathbf{C}[[h]]$ linear extension to a representation of $U_h(\mathcal{G})$ on $M_h \cong M[[h]]$. Define $R_h = (\rho_h \otimes \rho_h)\mathcal{R}_h$. Let σ be the transposition $\sigma(u \otimes v) = v \otimes u$ and $S_h = \sigma \circ R_h$.

Let $\{X_i\}$ be an orthonormal basis of $\mathcal{G}$ relative to the Killing form and

$$\mathbf{t} = \sum X_i \otimes X_i \in \mathcal{G} \otimes \mathcal{G} \tag{1.2}$$

be the polarized or "split" Casimir element. The equivalence theorem of Drinfeld [5], see §3 below, shows that there exists an $\mathcal{F}_h \in U(\mathcal{G})^{\otimes 2}[[h]]$, congruent to $1 \otimes 1$ mod h, such that

$$\mathcal{R}_h = \mathcal{F}_{h21} e^{h\mathbf{t}} \mathcal{F}_h^{-1}. \tag{1.3}$$

Thus $R_h = F_{h21}e^{ht}F_h^{-1}$ where $F_h = (\rho_h \otimes \rho_h)(\mathcal{F}_h)$ and $t = (\rho \otimes \rho)(\mathbf{t})$. The eigenvalues of S_h are $\lambda_{i,h} = e^{h\lambda_i}$ and $\mu_{j,h} = -e^{h\mu_j}$ where λ_i are the eigenvalues of t on $S^2(M)$ and μ_j the eigenvalues of t on $\wedge^2(M)$. Furthermore, the multiplicity of $\lambda_{i,h}$ is the same as that of λ_i and similarly for the μ_j's. To simplify notation we shall fix an eigenvalue $\lambda = \lambda_i$ and consider the case $\lambda_h = \lambda_{i,h} \equiv +1 \bmod h$. Obviously the case $\mu_{j,h} \equiv -1 \bmod h$ is completely analogous. Let

$$E'_{\lambda_h} = \mathrm{Im}(S_h - \lambda_h) = \bigoplus_{\lambda' \neq \lambda} hF_h(J_{\lambda'}) \oplus \bigoplus_{\mu} F_h(J_\mu); \tag{1.4}$$

where $J_{\lambda'}$ and J_μ are the eigenspaces of t on $S^2(M)$ and $\wedge^2(M)$ respectively. The factor of h in the first summand arises when there is more than one component J_λ because $\lambda_h - \lambda'_h = 0 \bmod h$. In this case $(M_h \otimes M_h)/E'_{\lambda_h}$ is not a free module and the resulting family of quadratic algebras is not flat.

To remedy this situation we add another component, $F_h(\mathrm{Im}(t-\lambda)|_{S^2(V)})$, to the definition of E'_h. Define the element

$$\mathbf{T}_h = \mathcal{F}_h \mathbf{t} \mathcal{F}_h^{-1} \in U(\mathcal{G})^{\otimes 2}[[h]], \tag{1.5}$$

and $T_h = \rho_h \otimes \rho_h(\mathbf{T}_h)$, and let λ be any eigenvalue of t on $S^2(M)$. Define

$$E_{\lambda_h} = \mathrm{Im}(\sigma \circ R_h - e^{h\lambda}) + \mathrm{Im}(T_h - \lambda). \tag{1.6}$$

At $h = 0$ we get

$$E_\lambda = \mathrm{Im}(\sigma - 1) + \mathrm{Im}(t - \lambda). \tag{1.7}$$

Recall that we are only explicitly considering one of the two possible cases. The other case when μ is an eigenvalue of t on the antisymmetric elements gives us

$$E_{\mu_h} = \mathrm{Im}(\sigma \circ R_h + e^{h\mu}) + \mathrm{Im}(T_h - \mu), \tag{1.8}$$

$$E_\mu = \mathrm{Im}(\sigma + 1) + \mathrm{Im}(t - \mu). \tag{1.9}$$

THEOREM 1.1: *Let M be an arbitrary, not necessarily irreducible, representation of a simple Lie algebra $\mathcal{G}$ over $\mathbf{C}$. Let $\mathbf{t}$ be the split Casimir element (1.2), λ, an eigenvalue of $\mathbf{t}$ on $S^2(M)$ and $\lambda_h = e^{h\lambda}$. Define E_{λ_h} and E_λ by (1.6) and (1.7) respectively; then the quadratic algebra $\{M_h, E_{\lambda_h}\}$ is formally flat and defines a quadratic deformation of $\{M, E_\lambda\}$. A similar statement is true for μ an eigenvalue of $\mathbf{t}$ on $\wedge^2(M)$ and the quadratic algebra $\{M_h, E_{\mu_h}\}$ defined by (1.8).*

Remarks:

(1) In definition (1.6) of E_{λ_h}, the second component, $\mathrm{Im}(T_h - \lambda_h)$, which we add to $E'_h = \mathrm{Im}(S_h - e^{h\lambda})$, is essential in order to insure flatness because h is not invertible in $\mathbf{C}[[h]]$. However, if we were to localize at $h = 0$ and pass to the field $(\mathbf{C}[[h]])_{(h)}$ of formal Laurent series, the resulting quadratic algebra would be defined by E'_h alone since $E'_h \otimes (\mathbf{C}[[h]])_{(h)} \cong E_h \otimes (\mathbf{C}[[h]])_{(h)}$.

(2) Suppose that the multiplicities of the eigenvalues of the quantum transposition, S_h, jump at $h = 0$. Then we redefine the space E as the limit of an eigenspace for $h \neq 0$. In the context of formal power series this is done by adding one additional relation given by $t - \lambda$. This explains the extra component $\mathrm{Im}(t - \lambda)$ in the definition (1.7). *In this case we do not get the standard Fade'ev–Reshetikhin–Takhtajan construction which would not be flat.*

(3) When h can be taken as a numerical parameter, as in the case when the transposition involves the standard Drinfeld–Jimbo R-matrix, the condition of formal flatness implies that the dimension of each homogeneous component is constant in h in some neighborhood of 0. This is so because formal flatness is equivalent to the defining ideal having a complementary $\mathbf{C}[[h]]$ submodule:

$$M_h^{\otimes n} \cong I_h^{(n)} \oplus J_h^{(n)}.$$

Consider the matrix of coefficients representing the generators of $J_h^{(n)} \cong M_h^{\otimes n}/I_h^{(n)}$ relative to a standard basis of $M_h^{\otimes n}$. It has a minor of maximal rank whose determinant is an invertible element of $\mathbf{C}[[h]]$. This implies that in some neighborhood of 0, the dimension of $M_h^{\otimes n}/I_h^{(n)}$ as a vector space over $\mathbf{C}$ is constant. However, one cannot expect that the neighborhood can be chosen uniformly for all homogeneous components since the dimension of the nth homogeneous component is directly related to the structure of the subalgebra of $\mathrm{End}(M^{\otimes n})$ generated by $n-1$ Yang–Baxter operators. For example, when the image of $\mathbf{t}$ in $\mathrm{End}(M^{\otimes 2})$ has two eigenvalues, $\lambda_\pm$, that subalgebra is a Hecke algebra with $n-1$ generators, X_i, satisfying $(X_i - q_+)(X_i - q_-) = 0$, where $q_\pm = e^{h\lambda_\pm}$. Such an algebra is semisimple except when the ratio of the eigenvalues is an $(n-1)$st root of unity. The exceptional values depend on the order of the tensor product.

(4) If we try to define a quadratic algebra quantization relative to the family $E_h = \mathrm{Im}(S_h - \lambda_{1,h}) \cap \cdots \cap \mathrm{Im}(S_h - \lambda_{k,h})$ complementary to the sum of two or more eigenspaces, then we run into problems as the following example shows.

Let M be the 4 dimensional representation of sl(2), then $M \otimes M$ is the sum of four irreducible subrepresentations of dimensions $7, 5, 3$, and 1, each of which is an eigenspace of the split Casimir operator. The subspace $\wedge^2 M$ is complementary to the 7 and 3 dimensional subrepresentations, and has the form $\mathrm{Im}(t-\lambda) \cap \mathrm{Im}(t-\lambda') = \mathrm{Im}((t-\lambda)(t-\lambda'))$. Suppose that the family of quadratic subspaces $E_h = \mathrm{Im}(S_h - \lambda_h) \cap \mathrm{Im}(S_h - \lambda'_h)$ extending $E_0 = \wedge^2 M$ defined a quadratic algebra deformation. This would give a quantization of the classical Drinfeld–Jimbo R-matrix bracket

$$\{f, g\} = X_+ f X_- g - X_+ g X_- f \tag{1.10}$$

on the algebra of polynomials on M. However, it is easy to check explicitly that this bracket doesn't satisfy the Jacobi identity. If the Drinfeld–Jimbo quantum R-matrix for $U_h(\mathrm{sl}(2))$ defined a quadratic algebra deformation $\{M_h, E_h\}$, the bracket (1.10) would be the leading term of the commutator of the product in the deformation and therefore would satisfy the Jacobi identity.

This shows that the construction will not, in general, work when we consider more than one eigenvalue; however, what actually goes wrong has not really been explained. We will return to this example in the remarks at the end of §3, when the cause of the problem will be clearer.

The proof of the theorem is given in section 3. Rather than prove the theorem directly we define an "equivalent" quadratic algebra in a category with nonassociative tensor product. In the next section we extend the definition of quadratic algebras to this non-associative setting.

2. Quasi-associative tensor algebras

We assume that the reader is familiar with the basic theory of monoidal categories, but repeat the following definition to fix the terminology. For more details see [11].

Given two monoidal categories, $\mathcal{C}, \mathcal{D}$, a functor $\mathcal{E}\colon \mathcal{C} \to \mathcal{D}$ is called a *monoidal functor* if it satisfies the following conditions:

(1) The image of the identity in $\mathcal{C}$ is equivalent to the identity in $\mathcal{D}$, $\mathcal{E}(1_{\mathcal{C}}) \simeq 1_{\mathcal{D}}$.
(2) There is a natural equivalence of functors $\mathcal{E}(X \otimes_{\mathcal{C}} Y) \overset{f_{X,Y}}{\to} \mathcal{E}(X) \otimes_{\mathcal{D}} \mathcal{E}(Y)$.

(3) The associativity constraint, $a_{X,Y,Z}$, of $\mathcal{C}$ is transformed into the associativity constraint of $\mathcal{D}$. The diagram

$$\begin{array}{ccc}
\mathcal{E}((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) & \xrightarrow{\mathcal{E}(a_{X,Y,Z})} & \mathcal{E}(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z)) \\
{\scriptstyle (f_{X,Y} \otimes \mathrm{id})(f_{X\otimes Y,Z})}\downarrow & & \downarrow{\scriptstyle (\mathrm{id} \otimes f_{Y,Z}) f_{X,Y\otimes Z}} \\
(\mathcal{E}(X) \otimes_{\mathcal{D}} \mathcal{E}(Y)) \otimes_{\mathcal{D}} \mathcal{E}(Z) & \xrightarrow{a_{\mathcal{E}(X),\mathcal{E}(Y),\mathcal{E}(Z)}} & \mathcal{E}(X) \otimes_{\mathcal{D}} (\mathcal{E}(Y) \otimes_{\mathcal{D}} \mathcal{E}(Z))
\end{array}$$

is commutative.

Dropping condition (3) defines what we shall call a **multiplicative functor** between monoidal categories.

Let Mod_A be the category of modules over a commutative algebra A with the monoidal structure given by the standard tensor product of A modules, $M \otimes_A N$. We say that a monoidal category, $\mathcal{C}$, is **fibered over** Mod_A when there is given a multiplicative functor, $\mathcal{T}$, from $\mathcal{C}$ to the category Mod_A. The associativity constraint induces in the natural way a transformation $(\mathcal{T}(X) \otimes \mathcal{T}(Y)) \otimes \mathcal{T}(Z) \to \mathcal{T}(X) \otimes (\mathcal{T}(Y) \otimes \mathcal{T}(Z))$ given by the formula,

$$\tilde{a}_{X,Y,Z} = (\mathrm{id} \otimes f_{Y,Z}) \circ f_{X,Y\otimes Z} \circ \mathcal{T}(a_{X,Y,Z}) \circ f^{-1}_{X\otimes Y,Z} \circ (f^{-1}_{X,Y} \otimes \mathrm{id}). \tag{2.1}$$

The functor $\mathcal{T}$ is not necessarily monoidal because the morphism $\tilde{a}_{X,Y,Z}$ is not necessarily the associativity constraint of the category Mod_A. Let X be an object of $\mathcal{C}$ and $M = \mathcal{T}(X)$. The following construction was first introduced by Markl and Stasheff, [10]. For convenience, we replace the explicit placement of parentheses in the non-associative tensor product with subscripts indicating symbolically their position. By an "n-fold bracketing" we mean an expression with n $\bullet$'s and $n-2$ parentheses describing one of the possible non-associative products of n elements, that is, one of the possible sequences of $n-1$ binary operations. Let $M_v^{\otimes n}$ be the non-associative tensor product corresponding to the n-fold bracketing v. For example

$$M^{\otimes 3}_{(\bullet\bullet)\bullet} = (M \otimes M) \otimes M.$$

It will be convenient to distinguish one bracketing, $[n]$, with the placement of parentheses expanded from the left:

$$[3] = (\bullet\bullet)\bullet, \quad [4] = ((\bullet\bullet)\bullet)\bullet, \quad \text{and so on.}$$

The composite associativity operators

$$\tilde{a}_{v',v}\colon M_v^{\otimes p} \to M_{v'}^{\otimes p} \tag{2.2}$$

can be constructed from products of the basic $\tilde{a}_{U,V,W}$ (where U, V, W are tensor products of X). These are all A module isomorphisms.

The n-fold bracketings form the vertices of a convex polytope K_n in $\mathbf{R}^{n-2}$, called the **Stasheff associahedron**. The edges correspond to the basic associativity constraints and the 2 cells are pentagons and rectangles. A composite associativity is represented by an edge path. The MacLane coherence theorem says that the composite associativity between the spaces $M_v^{\otimes n}$ and $M_{v'}^{\otimes n}$ is independent of the decomposition into a product of basic associativities. This fact follows from the simple connectivity of the 2 skeleton of K_n.

Let $\mathrm{Fr}(M)$ be the free non-associative tensor algebra on M considered as a module over A, and let $\otimes$ denote the product. $\mathrm{Fr}(M)$ is a graded A-algebra with components

$$\begin{aligned}
\mathrm{Fr}^0(M) &= A,\\
\mathrm{Fr}^1(M) &= M,\\
\mathrm{Fr}^2(M) &= M^{\otimes 2},\\
\mathrm{Fr}^3(M) &\cong (M\otimes M)\otimes M \quad \oplus \quad M\otimes(M\otimes M)\\
&= M^{\otimes 3}_{\bullet(\bullet\bullet)} \oplus M^{\otimes 3}_{(\bullet\bullet)\bullet},\\
\mathrm{Fr}^4(M) &\cong M^{\otimes 4}_{\bullet(\bullet(\bullet\bullet))} \oplus M^{\otimes 4}_{(\bullet\bullet)(\bullet\bullet)} \oplus M^{\otimes 4}_{\bullet((\bullet\bullet)\bullet)} \oplus M^{\otimes 4}_{(\bullet(\bullet\bullet))\bullet} \oplus M^{\otimes 4}_{((\bullet\bullet)\bullet)\bullet}, \text{ and so on.}
\end{aligned}$$

Consider the ideal $J \in \mathrm{Fr}(M)$ with nth graded component generated by the terms

$$x_{v'} - \tilde{a}_{v',v}(x_v) \quad \text{where} \quad x_v \in M_v^{\otimes n} \quad \text{and} \quad x_{v'} \in M_{v'}^{\otimes n}. \tag{2.3}$$

Define

$$M^{\odot n} = \mathrm{Fr}^n(M)/J^n \quad \text{and} \quad \bigodot M = \bigoplus M^{\odot n}. \tag{2.4}$$

PROPOSITION 2.1:

(1) *The nth graded component, $M^{\odot n}$, of $\bigodot M$ is naturally isomorphic to $M_{[n]}^{\otimes n}$, where $[n]$ is the distinguished vertex in K_n defined above with bracketings expanded from the left.*

(2) *The nonassociative multiplication* $\otimes$ *on* $\mathrm{Fr}(M)$ *induces on* $\bigodot M$ *a multiplication* $\odot$ *which is* **quasi-associative** *in the sense*

$$\odot_v[x_1 \otimes \cdots \otimes x_n] = \odot_{v'}[a_{v',v}(x_1 \otimes \cdots \otimes x_n)]$$

where $\odot_v$ *abbreviates the operation of taking a product of* n *terms in accordance with the bracketing* v.

The proof is straightforward; for details see the paper of Markl and Stasheff, [10]. Suppose that we have $Y \xrightarrow{j} X \otimes X$ in $\mathcal{C}$, for which $\mathcal{T}(Y) \xrightarrow{\mathcal{T}(j)} \mathcal{T}(X) \otimes \mathcal{T}(X) = M \otimes M$ is an imbedding of A modules. Let $E = \tilde{f}(\mathcal{T}(Y)) \hookrightarrow M \otimes M$ and consider the ideal $\hat{I}_E$ of $\mathrm{Fr}(M)$ generated by E. Its nth graded component $\hat{I}_E^{(n)}$ is the sum of terms $\hat{I}_{E,i,v}^{(n)}$ in which E appears in position i of a tensor product of n terms bracketed according to the vertex $v \in K_n$. The commutativity of the diagram

$$\begin{array}{ccc} (Y \otimes X) \otimes X & \longrightarrow & ((X \otimes X) \otimes X) \otimes X \\ {\scriptstyle a_{Y,X,X}} \big\downarrow & & {\scriptstyle a_{X \otimes X, X, X}} \big\downarrow \\ Y \otimes (X \otimes X) & \longrightarrow & (X \otimes X) \otimes (X \otimes X) \end{array} \tag{2.5}$$

and similar ones with Y in the middle of the tensor product or at the right imply that the associativity operators acting on $\mathrm{Fr}^n(M)$ give well defined isomorphisms between the components $I_{E,i,v}^{(n)}$ and $I_{E,i,v'}^{(n)}$ for a fixed i (the position of the factor E) and distinct bracketings v, v'. Let $I_E \subset \bigodot M$ be the image of $\hat{I}_E$ under the quotient map from $\mathrm{Fr}(M)$ to $\bigodot M$. Just as in the associative case, there are precisely $n-1$ summands $I_{E,i}^{(n)}$ and the choice of a particular bracketing, $v \in K_n$ defines an isomorphism between $I_{E,i}^{(n)}$ and $I_{E,i,v}^{(n)}$. Define the **quasi-associative quadratic algebra**

$$\{M, E\}^{\odot} = \bigoplus M^{\odot n} / I_E^{(n)} \tag{2.6}$$

for $I_E^{(n)} = I_{E,1}^{(n)} + \cdots + I_{E,n-1}^{(n)}$, where

$$I_{E,i}^{(n)} = \{\textstyle\sum (\cdots (m_1 \cdots) \odot m_{i-1}) \odot u) \odot m_{i+1}) \cdots \odot m_n | u \in E, \quad m_j \in M\}.$$

This definition clearly reduces to the usual one when the associativity constraints $\tilde{a}_{v,v'}$ give the standard identification of the various n-fold tensor products of A modules.

Suppose that we have two monoidal categories, $\mathcal{C}, \mathcal{D}$, both fibered over Mod_A and a monoidal functor $\mathcal{E}\colon \mathcal{C} \to \mathcal{D}$ which is compatible with the fiberings $\mathcal{T}^1$ and $\mathcal{T}^2$, that is, the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathcal{E}} & \mathcal{D} \\ {\scriptstyle \mathcal{T}^1}\downarrow & & {\scriptstyle \mathcal{T}^2}\downarrow \\ \mathrm{Mod}_A & \xrightarrow{\mathrm{id}} & \mathrm{Mod}_A \end{array}$$

is commutative. There are three natural equivalences (condition 2 in the definition) which appear in this situation, $f_{X,Y}$ related to the functor $\mathcal{E}$, and the two $b^i_{X,Y}$ related to $\mathcal{T}^i$ for $i = 1, 2$. Consider the following diagram:

$$(2.7)\qquad \begin{array}{ccc} \mathcal{T}^1(Y \otimes_{\mathcal{C}} Z) = \mathcal{T}^2(\mathcal{E}(Y \otimes_{\mathcal{C}} Z)) & \xrightarrow{\mathcal{T}^2(f_{Y,Z})} & \mathcal{T}^2(\mathcal{E}(Y) \otimes_{\mathcal{D}} \mathcal{E}(Z)) \\ {\scriptstyle b^1_{Y,Z}}\downarrow & & {\scriptstyle b^2_{\mathcal{E}(Y),\mathcal{E}(Z)}}\downarrow \\ \mathcal{T}^1(Y) \otimes \mathcal{T}^1(Z) & \longrightarrow & \mathcal{T}^2(\mathcal{E}(Y) \otimes \mathcal{T}^2(\mathcal{E}(Z)) = \mathcal{T}^1(Y) \otimes \mathcal{T}^1(Z), \end{array}$$

where the A module isomorphism in the bottom row, which we shall denote by $\tilde{f}_{Y,Z}$, is defined so that the diagram commutes. The image of the associativity constraint from $\mathcal{C}$, $\tilde{a}^1_{X,Y,Z}$, is related to the image of the corresponding associativity constraint from $\mathcal{D}$, $\tilde{a}^2_{\mathcal{E}(X),\mathcal{E}(Y),\mathcal{E}(Z)}$, by

$$(2.8)\qquad (\mathrm{id} \otimes \tilde{f}_{Y,Z}) \circ \tilde{f}_{X,Y\otimes Z} \circ \tilde{a}^1_{X,Y,Z} \circ \tilde{f}^{-1}_{X\otimes Y,Z} \circ (\tilde{f}^{-1}_{X,Y} \otimes \mathrm{id}) = \tilde{a}^2_{\mathcal{E}(X),\mathcal{E}(Y),\mathcal{E}(Z)}.$$

Let $M = \mathcal{T}^1(X) = \mathcal{T}^2(\mathcal{E}(X))$ for X an object in $\mathcal{C}$. We can define two quasi-associative algebras $\bigodot_{\mathcal{C}} M$ and $\bigodot_{\mathcal{D}} M$, constructed from the images in Mod_A of the associativity constraint of $\mathcal{C}$ and $\mathcal{D}$ respectively. In order to compare the two we need the following fact: for any $v \in K_n$ there is a well-defined A module automorphism $\tilde{f}_v\colon M_v^{\otimes n} \to M_v^{\otimes n}$ such that for any pair of vertices the following diagram:

$$(2.9)\qquad \begin{array}{ccc} M_v^{\otimes n} & \xrightarrow{\tilde{f}_v} & M_v^{\otimes n} \\ {\scriptstyle \tilde{a}^1_{v,v'}}\downarrow & & {\scriptstyle \tilde{a}^2_{v,v'}}\downarrow \\ M_{v'}^{\otimes n} & \xrightarrow{\tilde{f}_{v'}} & M_{v'}^{\otimes n} \end{array}$$

commutes. This follows from equation (2.8) and the fact that in any composition of elementary associativities the intermediate $\tilde{f}_{U,V}$ cancel. Let $J_{\mathcal{C}}$ be the ideal of $\mathrm{Fr}(M)$ defining $\bigodot_{\mathcal{C}} M$ and $J_{\mathcal{D}}$ the ideal defining $\bigodot_{\mathcal{D}} M$. Define

$\tilde{f}$: $\mathrm{Fr}(M) \to \mathrm{Fr}(M)$ as the sum of the $\tilde{f}_v$. From the definition (2.4) and the commutative diagram we conclude that there is an A module automorphism of $\mathrm{Fr}^n(M)$ mapping the ideal $J^n_{\mathcal{C}}$ to the ideal $J^n_{\mathcal{D}}$ and hence an isomorphism of A modules between $\bigodot_{\mathcal{C}} M$ and $\bigodot_{\mathcal{D}} M$. Furthermore, if $Y \to X \otimes X$ induces an imbedding $\mathcal{T}(Y) \to T(X) \otimes \mathcal{T}(X)$, then diagram (2.5) and definition (2.6) show that $\tilde{f}$ induces an isomorphism between quadratic algebras. The preceding discussion can be summarized in the following proposition.

PROPOSITION 2.2: *Let $\mathcal{C} \xrightarrow{\mathcal{E}} \mathcal{D}$ be a monoidal functor compatible with fiberings $\mathcal{T}^1$ and $\mathcal{T}^2$ over the category of A modules. Let $M = \mathcal{T}^1(X) = \mathcal{T}^2(\mathcal{E}(X))$. Assume that we have a morphism in $\mathcal{C}$, $Y \xrightarrow{j} X \otimes X$, and the corresponding morphism in $\mathcal{D}$, $\mathcal{E}(Y) \xrightarrow{\mathcal{E}(j)} \mathcal{E}(X \otimes X) \xrightarrow{f_{X,X}} \mathcal{E}(X) \otimes \mathcal{E}(X)$, which induce imbeddings $\bar{E}_Y = \mathcal{T}^1(Y) \hookrightarrow \mathcal{T}^1(X) \otimes \mathcal{T}^1(X) = M \otimes M$ and $E_Y = \tilde{f}_{X,X}(\bar{E}_Y) \hookrightarrow M \otimes M$. Let $\bar{I}_Y$ be the ideal in $\bigodot_{\mathcal{C}} M$ generated by $\bar{E}_Y$ and I_Y the ideal in $\bigodot_{\mathcal{D}} M$ generated by E_Y. Define quadratic algebras*

$$\{M, \bar{E}_Y\}^{\odot}_{\mathcal{C}} = \bigodot_{\mathcal{C}} M/\bar{I}_Y \quad \text{and} \quad \{M, E_Y\}^{\odot}_{\mathcal{D}} = \bigodot_{\mathcal{D}} M/I_Y. \tag{2.10}$$

Then the maps $\tilde{f}_v$ described above induce A module isomorphisms

$$\bigodot_{\mathcal{C}} M \cong \bigodot_{\mathcal{D}} M,$$
$$\{M, \bar{E}_Y\}^{\odot}_{\mathcal{C}} \cong \{M, E_Y\}^{\odot}_{\mathcal{D}}. \tag{2.11}$$

3. Quadratic algebras of modules over QUE algebras

We specialize the constructions of the previous section to the case of the monoidal category of modules over a QUE algebra of a simple Lie algebra. Let $U'_h(\mathcal{G}) = (U(\mathcal{G})[[h]], \mu_h, \Delta_h, \mathcal{R}_h)$ be the Drinfeld–Jimbo QUE algebra, for $\mathcal{G}$ a simple Lie algebra over $\mathbf{C}$, in the nonstandard presentation with undeformed multiplication as described in Section 1:

$$(U(\mathcal{G})[[h]], \quad \mu, \quad \Delta'_h = (\sigma_h^{-1} \otimes \sigma_h^{-1}) \circ \Delta_h \circ \sigma_h, \quad \mathcal{R}'_h = (\sigma_h^{-1} \otimes \sigma_h^{-1})(\mathcal{R}_h)).$$

The other QUE algebra we need to consider is Drinfeld's quasi-Hopf deformation which we denote $U(G)_\Phi$, [5],

$$U(G)_\Phi = (U(\mathcal{G})[[h]], \mu, \Delta, e^{ht}, \Phi_h).$$

In this case the multiplication and comultiplication, μ and Δ, are undeformed, the universal R matrix is a simple exponential, $e^{h\mathbf{t}}$, and the coassociativity operator, Φ_h, is derived from the transition between solutions of the Knizhnik–Zamalodchikov equation with different asymptotics; see [4]. This latter construction allows one to assume that Φ_h can be expressed as an exponential $e^{\varphi(h\mathbf{t}_{12},h\mathbf{t}_{23})}$ where $\varphi(a,b)$ is a series with values in the free $\mathbf{C}$[[h]] Lie algebra generated by a and b without constant or linear terms.

The categories $\mathcal{C}$ and $\mathcal{D}$ of $U(\mathcal{G})_\Phi$ and $U'_h(\mathcal{G})$ modules, respectively, are monoidal categories with fiberings over the category of $\mathbf{C}[[h]]$ modules given by the forgetful functors. In both categories the objects are $U(\mathcal{G})[[h]]$ modules and the morphisms are $U(\mathcal{G})[[h]]$ morphisms. In $\mathcal{C}$ the module structure on the tensor product is undeformed but the associativity constraint is non-trivial, given by the action of Φ_h. In $\mathcal{D}$ the module structure on the tensor product is defined by the deformed coproduct and the associativity constraint is trivial. Drinfeld has proved that there is an equivalence of braided monoidal categories $\mathcal{C}\xrightarrow{\mathcal{E}}\mathcal{D}$. $\mathcal{E}$ is the identity on objects and morphisms (hence compatible with the fibering). The natural transformation

$$\mathcal{E}(X\otimes_{\mathcal{C}} Y)\xrightarrow{f_{X,Y}}\mathcal{E}(X)\otimes_{\mathcal{D}}\mathcal{E}(Y)$$

is given by the action on $X\otimes Y$ of an invertible element of $\mathcal{F}_h\in U(\mathcal{G})^{\otimes 2}[[h]]$, which satisfies the following identities, [5]:

$$\mathcal{F}_h\Delta(b)\mathcal{F}_h^{-1}=\Delta'_h(b), \tag{3.1}$$

$$\mathcal{F}_{h21}e^{h\mathbf{t}}\mathcal{F}_h^{-1}=\mathcal{R}'_h, \tag{3.2}$$

$$\mathcal{F}_{h23}(\mathrm{id}\otimes\Delta)(\mathcal{F}_h)\Phi=\mathcal{F}_{h12}(\Delta\otimes\mathrm{id})(\mathcal{F}_h). \tag{3.3}$$

Equation (3.3) implies equation (2.8) for any triple X,Y,Z since the associativity constraint in $\mathcal{D}$ induces the identity morphism of $\mathbf{C}[[h]]$ modules on the right side of (2.8). Given (M,ρ_M), a representation of $\mathcal{G}$ on the $\mathbf{C}$ vector space M, we can define an object $X\in\mathcal{C}$ using the $\mathbf{C}[[h]]$ linear extension of ρ_M to represent $U(\mathcal{G})[[h]]$ on $M[[h]]$. The fibering over $\mathbf{C}[[h]]$ modules, given by the forgetful functor, sends X to $M[[h]]$. Let t be the representation of the polarized Casimir operator $\mathbf{t}$ acting on $M\otimes M$ and λ an eigenvalue of t. Define $\lambda_h=e^{h\lambda}$ and $\bar{E}_{\lambda_h}=\mathrm{Im}(\sigma\circ e^{h\mathbf{t}}-e^{h\lambda})+\mathrm{Im}(\mathbf{t}-\lambda)$. As above set $E_{\lambda_h}=\tilde{f}_{X,X}(\bar{E}_{\lambda_h})$. The equivalence of (braided) monoidal categories given by $\mathcal{F}$ and Proposition 2.2

imply that the quadratic algebras $\{M[[h]], \bar{E}_{\lambda_h}\}_{\mathcal{C}}^{\odot}$ and $\{M[[h]], E_{\lambda_h}\}_{\mathcal{D}}^{\odot}$ are isomorphic as $\mathbf{C}[[h]]$ modules. The latter quadratic algebra can be identified as the one appearing in Theorem 1.1. The operator $\tilde{f}_{X,X}$ is given by the representation of $\mathcal{F}_h$ on $M \otimes M[[h]]$, which we have denoted F_h. From the definition of the space E_{λ_h} we have

$$\begin{aligned} E_{\lambda_h} &= F_h(\mathrm{Im}(\sigma \circ e^{ht} - e^{h\lambda}) + \mathrm{Im}(t - \lambda)) \\ &= \mathrm{Im}(\sigma \circ (F_{h21} e^{ht} F_h^{-1}) - e^{h\lambda}) + \mathrm{Im}(F_h(t-\lambda)F_h^{-1}) \\ &= \mathrm{Im}(\sigma \circ R_h - e^{h\lambda}) + \mathrm{Im}(T_h - \lambda), \end{aligned}$$

which agrees with the definition of the subspace E_{λ_h} given in (1.6). Since the operator $\tilde{a}$ induced by the associativity constraint in $\mathcal{D}$ is the identity, we have the $\mathbf{C}[[h]]$ module isomorphism

$$\{M[[h]], \bar{E}_h\}_{\mathcal{C}}^{\odot} \cong \{M[[h]], E_h\}_{\mathcal{D}}^{\odot} \cong \{M[[h]], E_h\}. \tag{3.4}$$

In order to prove flatness as asserted in the theorem, it is enough to prove flatness for $\{M[[h]], \bar{E}_h\}_{\mathcal{C}}^{\odot}$, to which we now proceed.

Let $\bar{I}_{\lambda_h}$ be the ideal in $\bigodot_{\mathcal{D}} M[[h]]$ generated by $\bar{E}_{\lambda_h}$. The nth graded component $\bar{I}_h^{(n)}$ is the sum of $n-1$ terms

$$\bar{I}_{\lambda_h, i}^{(n)} = (M[[h]]^{\odot(i-1)} \odot \bar{E}_{\lambda_h}) \odot M[[h]]^{\odot(n-i-1)}.$$

Each of these terms may be represented in many different ways depending on the bracketing, but for each $1 \leq i \leq n-1$ the different representations are equivalent under quasi-associativity, as was pointed out in the discussion following Proposition 2.1.

Define the nth polarized Casimir by

$$\mathbf{t}^{(n)} = \Delta^{(n)}\mathbf{c} - \sum_{i=0}^{n-1} 1^{\otimes i} \otimes \mathbf{c} \otimes 1^{\otimes n-i-1} = 2 \sum \mathbf{t}_{ij},$$

where $\Delta^{(n)}$ is the $(n-1)$st iterated comultiplication with values in $U(\mathcal{G})^{\otimes n}$ and $\mathbf{t}_{ij}$ is the split Casimir appearing in positions i, j. Let $t^{(n)}$ be the image of $\mathbf{t}^{(n)}$ under the representation $\rho^{\otimes n}\colon U(\mathcal{G})^{\otimes n} \to \mathrm{End}(V^{\otimes n})$ and set $\lambda^{(n)} = n(n-1)\lambda$. Then define

$$J^{(n)} = \mathrm{Ker}(t^{(n)} - \lambda^{(n)})|_{S^n(M)}.$$

With these preliminaries, it is clear that once we have proven the following proposition we have proven Theorem 1.1.

PROPOSITION 3.1: *Let* $\hat{I}^{(n)}_{\lambda_h} \subset M[[h]]^{\otimes n}_{[n]} \cong M^{\otimes n}[[h]]$ *be the image of* $\bar{I}^{(n)}_{\lambda_h} \subset M[[h]]^{\odot n}$ *under the isomorphism* $(M[[h]])^{\odot n} \cong (M[[h]])^{\otimes n}_{[n]}$. *Let* $E_\lambda = \mathrm{Im}(\sigma - 1) + \mathrm{Im}(t - \lambda)$ *and* I_λ *the ideal generated by* E_λ *in* $\bigotimes M$. *For each* n

$$\hat{I}^{(n)}_{\lambda_h} = I^{(n)}_\lambda[[h]] \quad \text{and} \quad M^{\otimes n} = I^{(n)}_\lambda \oplus J^{(n)}_\lambda. \tag{3.5}$$

Therefore we have the following splittings as $\mathbf{C}[[h]]$ *modules:*

$$(M[[h]])^{\otimes n}_{[n]} \cong \hat{I}^{(n)}_h \oplus J^{(n)}_\lambda[[h]]. \tag{3.6}$$

Proof: We use induction. For $n = 2$, by definition, $\hat{I}^{(2)}_h = \hat{E}_h = E_h = \mathrm{Im}(\sigma \circ e^{ht} - e^{h\lambda}) + \mathrm{Im}(t - \lambda)$. We have $\mathrm{Im}(\sigma \circ e^{ht} - e^{h\lambda}) = \mathrm{Im}((\sigma \circ e^{h(t-\lambda)} - 1)$. Writing $e^{h(t-\lambda)} - 1 = (t - \lambda)g(h, t, \lambda)$, we see immediately

$$\sigma \circ e^{h(t-\lambda)} - 1 = (e^{h(t-\lambda)} - 1) \circ \sigma + \sigma - 1 \equiv \sigma - 1 \quad \text{modulo} \quad \mathrm{Im}(t - \lambda),$$

and therefore

$$\hat{I}^{(2)}_{\lambda_h} = \mathrm{Im}(\sigma - 1) + \mathrm{Im}(t - \lambda) = I^{(2)}_\lambda[[h]].$$

Before proving the induction, we will prove

$$I^{(n)}_\lambda = \mathrm{Im}(t_1 - \lambda) + \sum_{1 \leq i \leq n-1} \mathrm{Im}(\sigma_i - 1), \tag{3.7}$$

where t_i, the endomorphism of $M^{\otimes n}$ given by t acting in positions $i, i+1$ and σ_i, is the transposition in positions $i, i+1$ of the n-fold tensor product.

It follows immediately from the definition of the ideal I_λ that

$$I^{(n)}_\lambda = \sum_{1 \leq i < n-1} (\mathrm{Im}(t_i - \lambda) + \mathrm{Im}(\sigma_i - 1)).$$

For any i, t_i is conjugate to t_1 by a suitable permutation. In general, for an arbitrary endomorphism p conjugated by an arbitrary automorphism ξ, one has the identity:

(3.8)

$$\mathrm{Im}(\xi p \xi^{-1}) = \mathrm{Im}(\xi p) = \mathrm{Im}((\xi - 1 + 1)p) \subset \mathrm{Im}((\xi - 1)p) + \mathrm{Im}(p) \subset \mathrm{Im}(\xi - 1) + \mathrm{Im}(p).$$

Applying this argument repeatedly, the case of $t_1 - \lambda$ conjugated by a product of σ_i's gives

$$\sum_{1 \leq i < n-1} \mathrm{Im}(t_i - \lambda) + \mathrm{Im}(\sigma_i - 1) = \mathrm{Im}(t_1 - \lambda) + \sum_{1 \leq i < n-1} \mathrm{Im}(\sigma_i - 1). \tag{3.9}$$

Suppose that we have proven the theorem for $n-1$. By definition

$$\bar{I}^{(n)}_{\lambda_h} = \bar{I}^{(n-1)}_{\lambda_h} \odot_C M[[h]] + (M[[h]])^{\odot(n-2)} \odot_C \bar{I}^{(2)}_{\lambda_h} \subset M[[h]]^{\odot n},$$

therefore, in $M[[h]]^{\otimes n}_{[n]}$,

$$\begin{aligned}\hat{I}^{(n)}_{\lambda_h} &= \hat{I}^{(n-1)}_{\lambda_h} \otimes M[[h]] + a_n\left[(M[[h]]^{\otimes(n-2)}_{[n-2]}) \otimes E_h\right]\\ &= (I^{(n-1)}_{\lambda} \otimes M)[[h]] + a_n\left[(M^{\otimes(n-2)} \otimes E)[[h]]\right],\end{aligned}$$

where a_n is the associativity operator for $\tilde{C}$ from $(M[[h]]^{\otimes(n-2)}_{[n-2]}) \otimes (M[[h]]^{\otimes 2})$ to $M[[h]]^{\otimes n}_{[n]}$. It is defined by the action of

$$\begin{aligned}(\Delta^{(n-2)} \otimes \mathrm{id} \otimes \mathrm{id})\Phi^{-1} &= (\Delta^{(n-2)} \otimes \mathrm{id} \otimes \mathrm{id}) e^{-\varphi(h\mathbf{t}_{12}, h\mathbf{t}_{23})}\\ &= e^{-\varphi(h(\mathbf{t}_{1,n-1}+\cdots+\mathbf{t}_{n-2,n-1}), h\mathbf{t}_{n-1,n})},\end{aligned}$$

where, according to the usual convention, $\mathbf{t}_{ij} \in U(\mathcal{G})^{\otimes n}$ for $1 \le i < j \le n$ means the element with $\mathbf{t}$ in positions i and j and 1 in the remaining positions. The first equation follows from the mulitplicativity of Δ and the second follows from iterating the relations

$$(\Delta \otimes \mathrm{id})\mathbf{t}_{12} = \mathbf{t}_{13} + \mathbf{t}_{23} \quad \text{and} \quad (\Delta \otimes \mathrm{id})\mathbf{t}_{23} = \mathbf{t}_{34}.$$

Since $\varphi(h\mathbf{t}_{12}, h\mathbf{t}_{23})$ is a function of commutators we can replace $\mathbf{t}_{ij}$ by $\mathbf{t}_{ij} - \lambda$. Therefore there exists a function g of h and $t_{i,n-1}$ where i runs from 1 to $n-2$, which is congruent to 1 modulo h (hence invertible), and such that

$$\begin{aligned}\tilde{a}_n(t_{n-1} - \lambda) &\equiv (t_{n-1} - \lambda)g \mod \sum \mathrm{Im}(t_{i,n-1} - \lambda)\\ &\equiv (t_{n-1} - \lambda)g \mod \mathrm{Im}(t_1 - \lambda) + \sum_{1 \le i \le n-2} \mathrm{Im}(\sigma_i - \lambda)\end{aligned} \tag{3.10}$$

and

$$\tilde{a}_n(\sigma_{n-1} - 1) \equiv \sigma_{n-1} - 1 \mod \mathrm{Im}(t_{n-1} - \lambda) + \sum \mathrm{Im}(t_{i,n-1} - \lambda). \tag{3.11}$$

Equations (3.10) and (3.11) together with the induction hypothesis imply

$$\begin{aligned}\hat{I}^{(n)}_{\lambda_h} &= \mathrm{Im}(t_1 - \lambda) + \sum_{1 \le i < n-2} \mathrm{Im}(\sigma_i - 1) + \tilde{a}_n[\mathrm{Im}(t_{n-1} - \lambda) + \mathrm{Im}(\sigma_{n-1} - 1)]\\ &= \mathrm{Im}(t_1 - \lambda) + \sum_{1 \le i < n-2} \mathrm{Im}(\sigma_i - 1) + \mathrm{Im}(t_{n-1} - \lambda) + \mathrm{Im}(\sigma_{n-1} - 1)\\ &= \mathrm{Im}(t_1 - \lambda) + \sum_{1 \le i < n-1} \mathrm{Im}(\sigma_i - 1)\\ &= I^{(n)}_{\lambda}[[h]].\end{aligned}$$

To complete the proof we need to show that $J_\lambda^{(n)}$ is a complementary subspace to $I_\lambda^{(n)}$ in the $\mathbf{C}$ vector space $M^{\otimes n}$. First of all it is clear that the subspace $S^n(M)$ of symmetric n tensors is a complement to $\sum_{1\le i\le n-1}\mathrm{Im}(\sigma_i - 1)$. By repeated application of (3.8) it is clear that $t^{(n)}$ and $n(n-1)t_1$ induce the same operator on the quotient $M^{\otimes n}/(\sum \mathrm{Im}(\sigma_i - 1))$. Thus

$$\mathrm{Im}(t_1-\lambda) + \sum_{1\le i\le n-1}\mathrm{Im}(\sigma_i - 1) = \mathrm{Im}(t^{(n)} - \lambda^{(n)}) + \sum_{1\le i\le n-1}\mathrm{Im}(\sigma_i - 1).$$

Since $S^n(M)$ is invariant under $t^{(n)}$ and the latter acts semisimply, we have

$$\begin{aligned} S^{(n)}(M) &= \mathrm{Im}(t^{(n)} - \lambda^{(n)})|_{S^n(M)} \oplus \mathrm{Ker}(t^{(n)} - \lambda^{(n)})|_{S^n(M)} \\ &= \mathrm{Im}(t^{(n)} - \lambda^{(n)})|_{S^n(M)} \oplus J_\lambda^{(n)}. \end{aligned}$$

Therefore

$$\begin{aligned} M^{\otimes n} &= \sum_{1\le i\le n-1}\mathrm{Im}(\sigma_i - 1) \oplus S^n(M) \\ &= \sum_{1\le i\le n-1}\mathrm{Im}(\sigma_i - 1) \oplus \mathrm{Im}(t^{(n)} - \lambda^{(n)})|_{S^n(M)} \oplus J_\lambda^{(n)} \\ &= (\mathrm{Im}(t_1-\lambda) + \sum_{1\le i\le n-1}\mathrm{Im}(\sigma_i - 1)) \oplus J_\lambda^{(n)} \\ &= I_\lambda^{(n)} \oplus J_\lambda^{(n)}. \end{aligned}$$

This proves our proposition and hence the main theorem for quantum spaces associated to the quantized enveloping algebras. ∎

4. Quantum semigroups

The situation for the quantum semigroups, function algebras which are quadratic algebras defined on $\mathrm{End}(M_h)$, is somewhat different since the relevant category in this case is the category of bimodules over $U(\mathcal{G})_\Phi$ and $U_h(\mathcal{G})$. The defining ideal comes from the braiding on bimodules

$$\begin{aligned} &X\otimes Y \xrightarrow{s_{X,Y}} Y\otimes X \\ &x\otimes y \mapsto \mathcal{R}_{h21}\cdot(y\otimes x)\cdot\mathcal{R}_{h21}^{-1}. \end{aligned}$$

The associativity constraint is given by

$$\begin{aligned} &(X\otimes Y)\otimes Z \xrightarrow{a_{X,Y,Z}} X\otimes(Y\otimes Z) \\ &(x\otimes y)\otimes z \mapsto \Phi_h\cdot x\otimes(y\otimes z)\cdot\Phi^{-1}. \end{aligned}$$

The initial, naive, definition of the defining subspace is

$$\begin{aligned} E'_h &= \mathrm{span}\{R_{h21}(y \otimes x)R_{h21}^{-1} - x \otimes y | x, y \in \mathrm{End}(M_h)\} \\ &= \mathrm{span}\{(\sigma \circ R_h)x \otimes y(R_h^{-1} \circ \sigma) - x \otimes y | x, y \in \mathrm{End}(M_h)\} \\ &= \mathrm{span}\{(\sigma \circ R_h)x \otimes y - x \otimes y(\sigma \circ R_h) | x, y \in \mathrm{End}(M_h)\} \\ &= \mathrm{Im}(\mathrm{ad}(\sigma \circ R_h)) \end{aligned}$$

where ad indicates commutator in $\mathrm{End}(M_h)$. As before, in general, this will not give a flat deformation, so we modify the definition of $I_h^{(2)}$ by adding $\mathrm{Im}(\mathrm{ad}(F_h t F_h^{-1}))$. The correct definitions are

$$\begin{aligned} E_{h,\mathrm{ad}} &= \mathrm{Im}(\mathrm{ad}(\sigma \circ R_h)) + \mathrm{Im}(\mathrm{ad}(F_h t F_h^{-1})) \\ E_{\mathrm{ad}} &= \mathrm{Im}(\mathrm{ad}(\sigma)) + \mathrm{Im}(\mathrm{ad}(t)). \end{aligned}$$

THEOREM 4.1 (Quantum semigroups): *The quadratic algebra $\{\mathrm{End}(M_h), E_{h,\mathrm{ad}}\}$ is formally flat and defines a quadratic deformation of $\{\mathrm{End}(M), E_{\mathrm{ad}}\}$. The undeformed algebra can be identified with the subalgebra of the symmetric algebra $S(\mathrm{End}(M))$ consisting of endomorphisms commuting with the split Casimir operator, where we use the natural imbedding $S^n(\mathrm{End}(M)) \hookrightarrow \otimes^n(\mathrm{End}(M)) \cong \mathrm{End}(M^{\otimes n})$.*

Once again we prove the theorem by considering the corresponding quasi-associative quadratic algebra which is a quotient of $\bigodot_C \mathrm{End}(M_h)$ by the ideal $\bar{I}_{h,\mathrm{ad}}$ defined from

$$\bar{E}_{h,\mathrm{ad}} = \mathrm{Im}(\mathrm{ad}(\sigma \circ e^{ht})) + \mathrm{Im}(\mathrm{ad}(t)).$$

The inductive definition of the nth graded component of $\bar{I}_{h,\mathrm{ad}}$ is

$$\bar{I}_{h,\mathrm{ad}}^{(n)} = \bar{I}_{h,\mathrm{ad}}^{(n-1)} \odot_C \mathrm{End}(M_h) + \mathrm{End}(M_h)^{\odot n-2} \odot_C \bar{E}_{h,\mathrm{ad}}.$$

Let $\hat{I}_{h,\mathrm{ad}}^{(n)}$ be the image of $\bar{I}_{h,\mathrm{ad}}^{(n)}$ under the $\mathbf{C}[[h]]$ isomorphism of $\mathrm{End}(M_h)^{\odot n}$ with $\mathrm{End}(M_h)_{[n]}^{\otimes n}$; then

$$\bar{I}_{h,\mathrm{ad}}^{(n)} = \bar{I}_{h,\mathrm{ad}}^{(n-1)} \otimes_C \mathrm{End}(M_h) + a_n[\mathrm{End}(M_h)^{\otimes(n-2)} \otimes_C E_{h,\mathrm{ad}}]$$

where the associativity constraint a_n is now given by *conjugation* by $(\Delta^{[n-2]} \otimes \mathrm{id}^{\otimes 2})\Phi$. Using the exponential expression for Φ gives

$$a_n = e^{\mathrm{ad}\, \varphi(h(\mathbf{t}_{1,n-1}+\cdots+\mathbf{t}_{n-2,n-1}), h\mathbf{t}_{n-1,n})}.$$

PROPOSITION 4.2:

$$\bar{I}^{(n)}_{h,\mathrm{ad}} = \mathrm{Im}(\mathrm{ad}(t_{12})) + \sum_{1\le i\le n-1} \mathrm{Im}(\mathrm{ad}(\sigma_i)) = I^{(n)}_{\mathrm{ad}}[[h]].$$

The proof of the proposition follows from the same arguments as in the proof of (3.5) together with the following simple lemma applied to products of $t_{i,j}$ and σ_i.

LEMMA 4.3: *Let $u, v, w \in \mathrm{End}(M)$; then*

$$[uv, w] = [u, vw] + [v, wu]$$

therefore

$$\mathrm{Im}(\mathrm{ad}(uv)) \subset \mathrm{Im}(\mathrm{ad}(u)) + \mathrm{Im}(\mathrm{ad}(v)).$$

Remark: Let us consider in the non-associative context the example discussed at the end of §1 of the 4 dimensional representation of (2). It is easy to see that if $\wedge^2 M = \mathrm{Im}(t-\lambda_1)(t-\lambda_2)$ then $E_h = \mathrm{Im}(\sigma\circ e^{ht} - e^{h\lambda_1})(\sigma\circ e^{ht} - e^{h\lambda_2}) = \wedge^2 M_h$. Thus in $M_h^{\otimes 2}$, E_h has a complementary $\mathbf{C}[[h]]$ submodule given by symmetric 2 tensors, $S^2 M_h$. However, we shall see that the subspace $E_h \otimes M_h + a[M_h \otimes E_h]$ in the triple tensor product $(M_h^{\otimes 3})_{(\bullet\bullet)\bullet}$ (as above, a is the associativity operator) doesn't have a complementary submodule. Modulo h (or at $h = 0$), we have the vectorspace decomposition

$$M^{\otimes 3} = S^3 M \oplus (\wedge^2 M \otimes M + M \otimes \wedge^2 M).$$

The dimension of $S^3 M$ is 20 and the dimension of the term in parentheses is 44, since $\wedge^2 M \otimes M \cap M \otimes \wedge^2 M$ equals $\wedge^3 M$. Now consider the $C[[h]]$ module $E_h \otimes M_h + a(M_h \otimes E_h)$, where, in this case, a has the form $1\otimes 1\otimes 1 + h^2 X_+ \wedge X_- \wedge H$ mod h^3. The action of $X_+ \wedge X_- \wedge H$ on an element of $\wedge^3 M$ gives a symmetric 3 tensor. A recursive argument shows that $\wedge^2 M_h \otimes M_h \cap a(M_h \otimes \wedge^2 M_h) = 0$ and $\wedge^2 M_h \otimes M_h + a(M_h \otimes \wedge^2 M_h)$ is a direct sum. Let L be a vectorspace complement to $\wedge^3 M$ in $\wedge^2 M \otimes M$ and P a vectorspace complement to $\wedge^3 M$ in $M \otimes \wedge^2 M$; then

$$\wedge^2 M_h \otimes M_h \oplus a[M_h \otimes \wedge^2 M_h] = L[[h]] \oplus \wedge^3 M[[h]] \oplus a(P[[h]]) \oplus h^2 N,$$

where, modulo h, the basis for $N \subset S^3M$ has the form

$$\{(X_+ \wedge X_- \wedge H)(e_i \wedge e_j \wedge e_k)\}$$

where $\{e_1, e_2, e_3, e_4\}$ is a basis for M. The term h^2N means that there is torsion in the quotient module

$$M_h^{\otimes 3}/[\wedge^2 M_h \otimes M_h + a(M_h \otimes \wedge^2 M_h)]$$

and thus the quadratic algebra is not a free $C[[h]]$ module as required for flatness. To see the relation to our remarks in Section 1, note that the element $X_+ \wedge X_- \wedge H$ is the Schouten bracket of the classical R-matrix and, if it does not vanish, then the associated bracket will not in general satisfy the Jacobi identity. In particular the Jacobi identity fails in the polynomial algebra of the 4 dimensional representation of sl(2).

5. Examples

1. In the case when $\mathcal{G} = \mathrm{sl}(n)$ and $M = \mathbf{C}^n$ is the fundamental representation, we have the standard Drinfeld–Jimbo R matrix

$$R_h = e^h \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + \sum (e^h - e^{-h}) \sum_{i>j} e_{ij} \otimes e_{ji}.$$

Set $\lambda_h = e^h$ and $\mu_h = -e^{-h}$. The symmetric and skew-symmetric subspaces of $M^{\otimes 2}$ are irreducible, therefore the appearance of the Casimir element does affect the "naive" definitions and we get for the undeformed algebras the usual symmetric and exterior algebras. In this case the associative quadratic deformations defined in section 3 above reduce to Manin's construction of quantum linear spaces associated to the Drinfeld–Jimbo R matrix. Similarly, for End(M),

$$I_{\mathrm{ad}} = \mathrm{Im}(\mathrm{ad}(\sigma)) \quad \text{and} \quad I_{h,\mathrm{ad}} = \mathrm{Im}(\mathrm{ad}(\sigma \circ R_h))$$

so we have Manin's construction of the quantum semi-group $\underline{\mathrm{end}}(M)$.

2. In the case $\mathcal{G} = \mathrm{so}(n)$ and the fundamental representation, the presence of the Casimir operator is essential. Here the second symmetric power, $S^2(M)$, is the sum of two irreducibles, the trivial one dimensional representation on the element $\gamma = \sum e_i \otimes e_i$ where $\{e_i\}$ is an orthonormal basis for M, and the other an

orthogonal complement, $S_0^2(M)$. Let π_- be the orthogonal projection on $\wedge^2 M$, π_0 the orthogonal projection on $S_0^2(M)$, and π_1 the orthogonal projection on γ, and $\pi_+ = \pi_0 + \pi_1$. Let $\lambda_-, \lambda_0, \lambda_1$ be the eigenvalues of t on the corresponding subspaces, so $t = \lambda_- \pi_- + \lambda_0 \pi_0 + \lambda_1 \pi_1$. Using our previous convention, λ_+ can be either λ_0 or λ_1. Then

$$\begin{aligned} E_{\lambda_0} &= \mathrm{Im}(t - \lambda_0) + \mathrm{Im}(\sigma - 1) \\ &= \mathrm{Im}((\lambda_- - \lambda_0)\pi_- + (\lambda_1 - \lambda_0)\pi_1) + \mathrm{Im}((-\pi_- + \pi_0 + \pi_1) - (\pi_- + \pi_0 + \pi_1)) \\ &= \mathrm{Im}(\pi_-) + \mathrm{Im}(\pi_1). \end{aligned}$$

We can consider the undeformed quadratic algebra as constructed in two stages, first taking the quotient relative to the ideal defined by $\mathrm{Im}(\pi_-)$, which gives us the usual symmetric algebra, and then passing to the quotient relative to the ideal in the symmetric algebra generated by $\mathrm{Im}(\pi_1)$. This gives the quotient of a polynomial algebra in n variables x_i modulo the ideal generated by the polynomial $g(x) = \sum x_i^2$, which is the algebra of polynomial functions on the cone $g(x) = 0$. Our construction gives a flat quadratic deformation of this algebra.

If we consider the other eigenvalue of t on the symmetric elements, $\lambda_+ = \lambda_1$, the subspace $E_{\lambda_1} = \mathrm{Im}(\pi_-) + \mathrm{Im}(\pi_0)$ has codimension 1 and for $n > 2$, $I_{\lambda_1}^{(n)} = M^{\otimes n}$. This gives us an algebra of dimension $n + 2$ with $\mathbf{C}$ basis $\{1, x_1, \ldots, x_n, z\}$ and the only nontrivial products $1x_i = x_i 1 = x_i$, $x_i^2 = z$ for all i.

Since t has a unique eigenvalue λ_- on the skew symmetric elements,

$$\mathrm{Im}(\sigma + 1) + \mathrm{Im}(t - \lambda_-) = \mathrm{Im}(\sigma + 1),$$

and $\otimes\{M, I_{\lambda_-}\} = \bigwedge M$.

When we consider the matrices $\mathrm{End}(M)$, the defining ideal for the undeformed quadratic algebra is generated by

$$\mathrm{Im}(\mathrm{ad}(t)) + \mathrm{Im}(\mathrm{ad}(\sigma)) = \mathrm{Im}(\mathrm{ad}(\sigma)) + \mathrm{Im}(\mathrm{ad}(\pi_0)).$$

If we construct the quadratic algebra in two stages we get first the symmetric algebra, which, as above, we identify with the polynomials in variables x_{ij} for $1 \le i, j \le n$. The second step is to quotient by the ideal generated by $\mathrm{Im}(\mathrm{ad}(\pi_0))$. Then

$$\pi_0 = \frac{1}{n} \sum e_{ij} \otimes e_{ij},$$

and

$$\mathrm{ad}(\pi_0)(x_{pq}x_{rs}) = \sum_{i,j} \delta_{jp}\delta_{jr}x_{ip}x_{is} - \sum \delta_{qi}\delta_{si}x_{pj}x_{rj}.$$

Therefore a basis of polynomials in $\mathrm{Im}(\mathrm{ad}(\pi_0))$ is given by

$$\sum_i x_{ij}x_{ik}, \sum_i x_{ji}x_{ki}, \quad \text{for } j \neq k \quad \text{and} \quad \sum_i x_{ij}x_{ij} - \sum_i x_{ki}x_{ki}.$$

The quotient is the function algebra of the conformal matrices (with arbitrary conformal factor, including 0). The quadratic deformation so defined is Manin's quantum conformal semigroup $\underline{\mathrm{end}}(M, g)$.

3. Similarly when $\mathcal{G} = \mathrm{sp}(n)$ and $M = \mathbf{C}^{2n}$ is the fundamental representation, we have the decomposition of M^2 into three irreducibles: $S^2(M)$, the trivial one dimensional representation on the invariant skew-symmetric element ω, and $\wedge_0^2(M)$, the complement to ω in $\wedge^2(M)$. It is easy to check that the split Casimir operator has distinct eigenvalues on the invariant subspaces. In this case we get deformations of the symmetric algebra, the exterior algebra modulo the ideal generated by the symplectic form, and a finite dimensional algebra of dimension $n+2$, analogous to the example for $\mathrm{so}(n)$.

4. In the general case, let M be a representation of a semisimple Lie algebra $\mathcal{G}$ and J_λ be the λ eigenspace of the split Casimir t in $S^2(M)$. $E_\lambda = \mathrm{Im}(t-\lambda)+\mathrm{Im}(\sigma-1)$ is the sum of eigensubspaces corresponding to the other eigenvalues of t in $S^2(M)$ and $\wedge^2(M)$. Then $M^{\otimes 2} = J \oplus E_\lambda$. Theorem 1.1 gives the flat deformation of the quadratic algebra $\bigotimes\{M, I\}$. The nth homogeneous component of this algebra will be isomorphic to the space $J^{(n)} = \bigcap_{i=1}^{n-1} J_{i,i-1} \subset M^{\otimes n}$.

In particular, let M have the highest weight α, J_α the component in $S^2(M)$ of the highest weight 2α and E_α the sum of all the other weight spaces. The split Casimir separates J_α since this space corresponds to the strictly maximal eigenvalue $\langle\alpha, \alpha\rangle$. It follows from a theorem of B. Kostant, see [8], that the quadratic algebra $\{M, E_\alpha\}$ is the algebra of functions on the orbit of the highest vector under action on M of the group G corresponding to the algebra $\mathcal{G}$. Its component in graded degree n is isomorphic as a G module to the irreducible subrepresentation of $M^{\otimes n}$ of the highest weight $n\alpha$ and Theorem 1.1 gives the deformation (or quantization) A_h of this algebra by the R-matrix Poisson bracket. This bracket is induced on the orbit by the Drinfeld–Jimbo R-matrix of the form

$$R = R_{DJ} = \frac{1}{2} \sum_{\alpha\in\Omega_+} X_\alpha \wedge X_{-\alpha} \in \wedge^2\mathcal{G}, \tag{4.1}$$

where $\{H_\alpha, X_\alpha, X_{-\alpha}\}$, $\alpha \in \Omega_+$, is the Cartan–Chevalley system in $\mathcal{G}$ and Ω_+ is the set of all positive roots of $\mathcal{G}$. Other proofs of the existence of quantization of the R-matrix Poisson bracket on the highest weight orbits and on the flag manifolds not using the Kostant theorem are given in [1] and [2].

5. When we consider the matrices End(M), for an arbitrary representation of a semisimple Lie algebra $\mathcal{G}$, the quantum semigroup constructed described in Theorem 3.2 is a deformation of the function algebra with defining equation

$$\sum_{kl} t_{ij,kl} x_{km} x_{ln} - t_{kl,mn} x_{ik} x_{jl} = 0,$$

where $t_{ij,kl}$ is the split Casimir acting on $M \otimes M$ and x_{ij} are the coordinate functions of End(M). This is the function algebra of the semigroup of endomorphisms a of M such that the product endomorphism $a \otimes a$ preserve the eigenspaces of the Casimir acting on $M \otimes M$.

6. The results of this paper can be deduced from the following general result.

Let V be a finite dimensional vector space and S a semisimple linear operator acting on $V \otimes V$. Then we have the following decomposition: $V \otimes V = \bigoplus_{k=1}^{m} I_k$, where I_k are eigenspaces corresponding to the distinct eigenvalues, $\lambda_1, \ldots, \lambda_m$, of S. Denote $J_k = \bigoplus_{i \neq k} I_i$, so $V \otimes V = I_k \oplus J_k$.

Denote by $A_2(S)$ the associative subalgebra in End$(V \otimes V)$ generated by S. It is a semisimple algebra. Let $A_n(S)$ be the associative subalgebra in End$(V^{\otimes n})$ generated by the operators S_i, $i = 1, \ldots, n-1$, where S_i denotes the operator on End$(V^{\otimes n})$ which coincides with S in the position $i, i+1$ and is identical in the other positions.

Let S_h be a deformation of the operator S, $S_0 = S$. Suppose that the algebra $A_2(S_h)$ has a constant dimension as a vector space, i.e. dim $A_2(S_h) =$ dim $A_2(S_0)$. Then the deformations of eigenvalues, $\lambda_{k,h}$, and the corresponding eigenspaces, $I_{k,h}$, are well defined, therefore the deformations of the subspaces $J_{k,h}$ are also well defined.

THEOREM: *If all the algebras $A_n(S_0)$, $n = 2, 3, \ldots$, are semisimple and* dim $A_n(S_h) =$ dim $A_n(S_0)$, *then $(V, J_{k,h})$ defines a flat deformation of the quadratic algebra (V, J_k) for all $k = 1, \ldots, m$. Moreover, in this case the quantum semigroup associated to S_h forms a flat deformation of the quantum semigroup associated to S.*

We shall give the proof of the theorem in a forthcoming paper.

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